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Wielandt's Subnormality Criterion and Linear Groups

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INTRODUCTION

In [14] Wielandt proved that a subgroup H of a finite group G is subnormal in G if and only if for each $g \in G$ and $h \in H$ there exists an integer m with $[g, {}_m h] \in H$. Several authors have investigated this criterion in other classes of groups, e.g., [3–6]. Here we consider linear groups, both finite and infinite.

We use the following notation. Let G be a group and H a subgroup of G. Write:

H asc G	if H is an ascendant subgroup of G,
H sn G	if H is a subnormal subgroup of G,
$H \trianglelefteq^m G$	if H is subnormal in G of depth at most m,
$G\,\mathbf{w}\,H$	if for each $g \in G$ and $h \in H$ there exists $m \in \mathbb{N}$ with $[g, {}_m h] \in H$,
$G\,\vert\mathbf{w}\,H$	if for each $h \in H$ there exists $m \in \mathbb{N}$ such that for all $g \in G$ we have $[g, {}_m h] \in H$,
$G\,\mathbf{w}_m\,H$	if for all $g \in G$ and $h \in H$ we have $[g, {}_m h] \in H$ and
$G\,\vert\mathbf{w}\vert\,H$	if there exists $m \in \mathbb{N}$ with $G\,\mathbf{w}_m\,H$.

I apologise for introducing yet another notation, but I intend the above to be more systematic. In particular the $\mathbf{w}$ notation is supposed to mirror a standard Engel notation (e.g., [8, p. 110]) with a W for Wielandt replacing the E for Engel. The following implications are trivial.

$$\begin{array}{ccccccc}
H \trianglelefteq^m G & \longrightarrow & H \text{ sn } G & & \Longrightarrow & & H \text{ asc } G \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
G\,\mathbf{w}_m\,H & \Longrightarrow & G\,|\mathbf{w}|\,H & \longrightarrow & G\,|\mathbf{w}\,H & \Longrightarrow & G\,\mathbf{w}\,H
\end{array}$$

The object is to produce partial converses of these implications.

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1.1 LEMMA. *There exists an integer-valued function $s(m, n)$ of m and n only such that if G is a finite linear group of degree n and if H is a subgroup of G with $G\,\mathbf{w}_m\,H$ then $H \lhd^{s(m,n)} G$.*

The bound in 1.1 depends on m and n. For example, $GL(2, \mathbb{C})$ contains a dihedral subgroup D_i of order 2^{i+1}. If H_i is generated by a non-central involution of D_i then H_i is subnormal on D_i of depth exactly i. Also if $W_i = C \wr K_i$, where $|C| = 2$ and K_i is elementary abelian of order 2^{i-1}, then $W_i\,\mathbf{w}_2\,K_i$ and the subnormal depth of K_i is exactly i. The lemma enables us to state our first theorem.

1.2 THEOREM. *Let G be a linear group of degree n, let H be a subgroup of G and denote the (Zariski) closure of H in G by $\bar{H}$. If $G\,\mathbf{w}_m\,H$ then $\bar{H} \lhd^{s(m,n)} G$.*

In fact we obtain this conclusion for certain other subgroups of G associated with H, see 4.2 and 4.3, but not for H itself. There exist linear groups G with subgroups H such that $G\,\mathbf{w}_2\,H$ and $H = N_G(H)$ and there exist finitely generated linear groups G with non-ascendant subgroups H satisfying $G\,\mathbf{w}\,H$, see 8.1 and 8.3, and in both cases there exist such G represented over $\mathbb{Z}$. The situation is little better if G is soluble. There is a metabelian linear group G with a subgroup H such that $G\,\mathbf{w}_2\,H$ and $H = N_G(H)$ and with a non-subnormal subgroup K such that $G\,\mathbf{w}_2\,K$ and $K \operatorname{asc} G$; see 8.2 and 8.7. All the above examples can be found with degree 2. However, our examples leave the possibility that one can deduce something about $\bar{H}$ from $G\,|\mathbf{w}\,H$ or even from $G\,\mathbf{w}\,H$ for arbitrary H. We also leave open the possibility that for finitely generated linear groups G we have $H \operatorname{asc} G$ (or even $H \operatorname{sn} G$) whenever $G\,\mathbf{w}_m\,H$. This is not possible if there exists an m-Engel group that is not locally nilpotent.

The soluble examples mentioned above require a relatively large representing field. Soluble groups over small ground-rings behave nicely, and then only the solubility restriction on H is required.

1.3 THEOREM. *Let G be a linear group of degree n and let H be a soluble-by-finite subgroup of G such that either* (i) *G or* (ii) *H is finitely generated.*

(a) *If $G\,\mathbf{w}\,H$ then $H \operatorname{asc} G$.*

(b) *If $G\,|\mathbf{w}\,H$ then $H \operatorname{sn} G$.*

(c) *If $G\,\mathbf{w}_m\,H$ then $H \lhd^d G$, where d is determined by m, n and G in Case* (i), *m, n and H in Case* (ii).

In fact we prove 1.3 under somewhat weaker assumptions; see 6.2. It is not possible to deduce $H \operatorname{sn} G$ from $G\,\mathbf{w}\,H$ even if both G and H are finitely generated and soluble; see 8.5. Also the infinite dihedral groups shows that if H sn G the depth of H cannot be bounded in terms of n and G in Case (i), H in Case (ii) only; see 8.8.

1.4 THEOREM. *Let G be a subgroup of $GL(n, \mathbb{Q})$ and let H be a soluble-by-finite subgroup of G.*

(a) *If G $\mathbf{w}$ H then H* asc *G.*

(b) *If $G \,|\mathbf{w}\, H$ then H* sn *G.*

(c) *If $G\, \mathbf{w}_m\, H$ then $H \lhd^d G$, where d is determined by m and n only.*

Examples 8.5 and 8.8 show that in 1.4, G $\mathbf{w}$ H does not imply H sn G and d must depend on m even for a fixed G. Using 1.4 one can extend part of [4]. Following mainly [7] we use the following notation. $\mathfrak{S}_0$ is the class of groups with a series of finite length whose factors are either torsion-free abelian of finite rank or torsion abelian with each primary component of finite rank. $\mathfrak{S}_1$ is defined as $\mathfrak{S}_0$ except the torsion abelian factors in the series are also required to involve only a finite number of primes. $\mathfrak{S}_2$ is the class of soluble groups with a series of finite length whose factors satisfy either the maximal or the minimal condition on subgroups $\hat{\mathfrak{S}}$ is the class of soluble groups of finite rank. $\mathfrak{S}_t$ is the class of torsion-free-by-finite $\mathfrak{S}_0$-groups. $\mathfrak{F}$ is the class of finite groups. Then

$$\mathfrak{S}_0 \supset \hat{\mathfrak{S}} \supset \mathfrak{S}_1 \supset (\mathfrak{S}_2 \cup \mathfrak{S}_t).$$

1.5 COROLLARY. *Let $G \in \mathfrak{S}_0\mathfrak{F}$ and let H be a subgroup of G.*

(a) *If G $\mathbf{w}$ H then H* asc *G.*

If $G \in \mathfrak{S}_1\mathfrak{F}$ then

(b) *$G \,|\mathbf{w}\, H$ implies H* sn *G*

and

(c) *$G\, \mathbf{w}_m\, H$ implies $H \lhd^d G$, where d depends only on m and G.*

McCaughan and McDougall prove 1.5(b) under the stronger hypothesis that $G \in \mathfrak{S}_2\mathfrak{F}$. There exists $G \in \hat{\mathfrak{S}}$ with a non-subnormal subgroup H satisfying $G \,|\mathbf{w}\, H$; see 8.6. Also the infinite dihadral group lies in $\mathfrak{S}_2 \cap \mathfrak{S}_t$ so we cannot delete the dependence on m in Part (c). The following is an immediate corollary of 1.4 and [10, 1.1].

1.6 COROLLARY. *Let $X \in \mathfrak{S}_t\mathfrak{F}$. Let G be a subgroup of the holomorph of X and let H be a soluble-by-finite subgroup of G.*

(a) *If G $\mathbf{w}$ H then H* asc *G.*

(b) *If* $G\,|\mathbf{w}\,H$ *then* H sn G.

(c) *If* $G\,\mathbf{w}_m\,H$ *then* $H \lhd^d G$, *where* d *depends on* m *and* X *only.*

There is not much scope for generalising 1.6 since if X is the direct product of two Prüfer p^∞-groups then $\operatorname{Aut} X$ is isomorphic to $GL(2, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers. $GL(2, \mathbb{Z}_p)$ contains copies of examples 8.1, 8.2, 8.3, 8.7, 8.8 and, if $p \neq 2$, 8.5.

2. Groupring Lemmas

If G is any group and m is any positive integer, set

$$\mathfrak{g}_m = \sum_{g \in G} (g-1)^m \,\mathbb{Z}G,$$

$\mathbb{Z}G$ denoting the integral groupring of G. Write $\mathfrak{g}$ for $\mathfrak{g}_1$.

2.1. *If* G *is finite of order* t *we have* $\mathfrak{g}^{t(t-1)(m-1)} \subseteq \mathfrak{g}_m$.

This is a weak form of [11, Lemma 3].

2.2 (Gruenberg). *Set*

$$\begin{aligned} m^* &= \prod_{i=0}^{m=2} (m-i)^{2^i} && \textit{if} \quad m \geqslant 2 \\ &= 1 && \textit{if} \quad m = 1. \end{aligned}$$

If G *is abelian then* $m^* \cdot \mathfrak{g}^{2^{m-1}} \subseteq \mathfrak{g}_m$.

See [2, Lemma 4.1].

2.3. *If* G *is a* d*-generator abelian group then* $\mathfrak{g}^{dm} \subseteq \mathfrak{g}_m$.

Proof. This can be proved by direct calculation. Alternatively consider the split extension W of $A = \mathbb{Z}G/\mathfrak{g}_m$ by G. For each element g of a fixed generating set of G the group $\langle g \rangle A$ is normal in W and nilpotent of class at most m. By Fitting's Theorem W is nilpotent of class at most dm and the result follows.

3. Group Theoretical Lemmas

For 3.1 to 3.5 below let G be a group, A a normal subgroup of G and H a subgroup of G such that $G = AH$.

3.1. *If A is finite and G **w** H then H* sn *G.*

This follows at once from Wielandt's Theorem applied to $G/C_H(A)$.

3.2. *Suppose that A is abelian and G/A is finite. If G **w** H then H* asc *G. If $G \,|\, \mathbf{w}\, H$ then H* sn *G.*

Proof. $A \cap H \lhd G$, so we may assume that $A \cap H = \langle 1 \rangle$. If G **w** H then A **e** G and if $G\,|\,\mathbf{w}\, H$ then $A \,|\,\mathbf{e}\, G$. By [2, 2.2], in the first case $A \subseteq \zeta_\omega(G)$ and in the second $A \subseteq \zeta_d(G)$ for some finite d, where $\zeta_i(G)$ denotes the ith term of the upper central series of G. The result follows. (3.2 can also be derived from 2.1.)

3.3. *Suppose A is a subgroup of a vector space V of finite dimension n such that the action of H on A extends to one of H on V. If $A \cap H = \langle 1 \rangle$ and $G \mathbf{w} H$ then $H <^n G$.*

Proof. We may assume that A generates V as vector space. Since $A \cap H = \langle 1 \rangle$, the group H acts unipotently on V. Thus $[V, {}_nH] = \langle 1 \rangle$ and $H <^n G$.

3.4. *Suppose A is a direct product of n Prüfer p^∞-groups. If G **w** H then $H \lhd^\omega G$ and if $G\,|\,\mathbf{w}\, H$ then $H \lhd_n G$.*

Proof. $A \cap H < G$, so we may assume $A \cap H = \langle 1 \rangle$. We may also factor out by $C_H(A)$ and assume that H acts faithfully on A. If $A_i = \{a \in A : a^{p^i} = 1\}$ then A_i is finite, H sn A_iH by 3.1 and $A = \bigcup_i A_i \subseteq \zeta_\omega(G)$. Thus $H \lhd^\omega G$. Now assume $G\,|\,\mathbf{w}\, H$. After a choice of basis of A we may regard H as a subgroup of $GL(n, \mathbb{Z}_p)$. If $h \in H$ there exists m such that $[A, {}_mh] = \langle 1 \rangle$. Then $(h-1)^m = 0$ in the matrix ring $(\mathbb{Z}_p)_n$ and H is unipotent. Since $\mathbb{Z}_p$ is a principal ideal domain there exists $x \in GL(n, \mathbb{Z}_p)$ such that H^x is unitriangular. Thus $[A, {}_nH] = \langle 1 \rangle$ and $H \lhd^n G$.

3.5. *If A is an abelian $\mathfrak{S}_0$-group and $G \mathbf{w} H$ then H* asc *G, in fact. $H \lhd^{\omega+r} G$ for some finite r.*

Proof. $\mathfrak{S}_0$ is quotient closed so we may assume that $A \cap H = \langle 1 \rangle$. If $q \in \mathbb{N}$ let $A_q = \{a \in A : a^q = 1\}$. Then A_q is finite so by 3.1 we have H sn A_qH and $A_q \subseteq \zeta_\omega(A_qH)$. Let $T = \bigcup A_q$ be the torsion subgroup of A. Then $T \subseteq \zeta_\omega(TH)$ and $H <^\omega TH$. Also A/T is torsion-free of finite rank and $A \cap TH = T$, so TH/T sn G/T by 3.3 and the result follows.

A simple induction using 3.2 for the initial case and 3.5 for the induction step yields

3. COROLLARY. *If* $G \in \mathfrak{S}_0\mathfrak{A}\mathfrak{F}$ *and* $G \,\mathbf{w}\, H$ *then* H asc G.

Corollary 1.5(a) is a special case of this. Reference [4, Theorem 6] follows in a similar way. The remainder of 1.5 is less trivial.

4. ARBITRARY LINEAR GROUPS

4.1 *Proof of* 1.1. We may assume that G is a subgroup of $GL(n, F)$, where $n > 1$ and F is an algebraically closed field. Suppose first that G is soluble. By Mal'cev's Theorem [8, 3.6] G has a triangularizable normal subgroup T of index bounded by a function $\mu(n)$ of n only. Let U be the unipotent radical of T. Then U is nilpotent of class at most $n-1$; let Z denote its centre and set $K = H \cap T$.

Now $H \cap Z$ is normal in HZ. Also $G \,\mathbf{w}_m\, H$ implies that $[z, {}_m h] \in H \cap Z$ for all $z \in Z$ and $h \in H$. By 2.1 (or 2.2, if you prefer) the hypercentre of $KZ/(H \cap Z)$ contains $Z/(H \cap Z)$. If char $F = p > 0$ then Z is a p-group and $K/C_K(Z)$ is a p'-group. Thus $[Z, K] \subseteq H \cap Z$. If char $F = 0$ then $Z = \langle 1 \rangle$. Since $K \lhd H$ we have $K \lhd HZ$ in both cases. Now apply 2.1 to the action of H/K on KZ/K. We obtain $[Z, {}_t H] \subseteq K \cap Z \subseteq H$, where

$$t = \mu(n)(\mu(n) - 1)(m - 1).$$

Thus $H \lhd^t HZ$. A simple induction on the class of U yields $H \lhd^{t(n-1)} HU$. Since T/U is abelian 2.1 applied to the action of H/K on $T/(T \cap HU)$ shows that $[T, {}_t H] \subseteq T \cap HU$ and $HU \lhd^t HT$. Finally $HT \lhd^{\mu(n)} G$ since $(G:T) \leqslant \mu(n)$ and we have shown that $H \lhd^{tn+\mu(n)} G$.

Now consider the insoluble case. We know H sn G so suppose

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_s = G.$$

By a lemma of Kargapolov [8, 9.29] there exists a function $k(n)$ of n only such that at most $k(n)$ of the factors H_i/H_{i-1} are insoluble. Suppose H_i/H_{i-1} is soluble for $r < i \leqslant s$. Then there exists $L \lhd H_s$ with $L \subseteq H_r$ and H_s/L soluble. By a theorem of P. Hall (or [8, 9.20]) there exists a soluble subgroup S of G with $LS = H_s$. By the soluble case $(H_r \cap S) \lhd^{tn+\mu(n)} (H_s \cap S)$. But $L \lhd H_s$ and $H_i = L(H_i \cap S)$ for $i = r, s$, so $H_r \lhd^{tn+\mu(n)} H_s$. We have now shown that $H \lhd^{s(m,n)} G$, where

$$s(m, n) = (k(n) + 1)(\mu(n)(\mu(n) - 1)(m - 1)n + \mu(n)) + k(n).$$

4.2. *Let R be a finitely generated integral domain, let G be a subgroup of*

$GL(n, R)$ and let H be a subgroup of G with $G\,\mathbf{w}_m H$. Then $\tilde{H} \lhd^{s(m,n)} G$, where

$$\tilde{H} = \bigcap_{\mathfrak{m}} H(G \cap (1 + (\mathfrak{m})_n)),$$

the intersection being taken over all the maximal ideals $\mathfrak{m}$ of R.

Here and throughout $s(m, n)$ is any function satisfying 1.1; $(\mathfrak{m})_n$ is the n by n matrix ring over m.

Proof. Reducing matrix entries modulo $\mathfrak{m}$ determines a homomorphism $\phi_{\mathfrak{m}}$ of G into $GL(n, R/\mathfrak{m})$. By 4.1 we have $H\phi_{\mathfrak{m}} \lhd^{s(m,n)} G\phi_{\mathfrak{m}}$. Thus $H(G \cap (1 + (\mathfrak{m})_n)) \lhd^{s(m,n)} G$ and the conclusion follows.

If $X \subseteq GL(n, F)$ we denote by $\mathscr{A}_F(X)$ the intersection of all the Zariski closed subgroups of $GL(n, F)$ containing X.

4.3. *Let F be a field, G a subgroup of $GL(n, F)$ and H a subgroup of G with $G\,\mathbf{w}_m H$. Set $L = \bigcup_Y (Y \cap \mathscr{A}_F(H \cap Y))$, where the union runs over all the finitely generated subgroups Y of G. Then $L \lhd^{s(m,n)} G$.*

L necessarily is a subgroup of $G \cap \mathscr{A}_F(H)$ and frequently is much smaller. For example if G is periodic then G is locally finite by Schur's Theorem and $L = H$. Trivially $H \subseteq L$.

Proof. Let Y be any finitely generated subgroup of G. Then $Y \subseteq GL(n, R)$ for some finitely generated subring R of F. In the obvious notation 4.2 yields $\widetilde{H \cap Y} \lhd^{s(m,n)} Y$. But

$$H \cap Y \subseteq \widetilde{H \cap Y} \subseteq Y \cap \mathscr{A}_R(H \cap Y) = Y \cap \mathscr{A}_F(H \cap Y)$$

by Lemma 1 of [9]. Thus

$$Y \cap \mathscr{A}_F(\widetilde{H \cap Y}) = Y \cap \mathscr{A}_F(H \cap Y)$$

and so $Y \cap \mathscr{A}_F(H \cap Y) \lhd^{s(m,n)} Y$ by [8, 5.10]. Therefore

$$\begin{aligned}[G, {}_{s(m,n)}L] &\subseteq \bigcup_Y [Y, {}_{s(m,n)}Y \cap \mathscr{A}_F(H \cap Y)] \\ &\subseteq \bigcup_Y (Y \cap \mathscr{A}_F(H \cap Y)) = L.\end{aligned}$$

Consequently $L \lhd^{s(m,n)} G$ as required.

4.4 *Proof of* 1.2. Assume the notation of 4.3. Since $H \subseteq L \subseteq G \cap \mathscr{A}_F(H)$ we have $G \cap \mathscr{A}_F(L) = G \cap \mathscr{A}_F(H)$. The result now follows from 4.3 and [8, 5.10].

5. Normal Closure

5.1 Proposition. *Let G be a linear group of degree n and suppose that H is a subgroup of G with a soluble normal subgroup S of finite index. If $G \mathbf{w} H$ then the normal closure $K = H^G$ has a soluble normal subgroup T of finite index such that $(K : T)$ is bounded by a function of n and $(H : S)$ only.*

We break the proof into three pieces.

Assume G is finite. By Wielandt's Theorem H is subnormal in G and a simple induction shows that every composition factor of K is isomorphic to a composition factor of H. Thus $(H : S)$ bounds the orders of the non-abelian composition factors of K. A non-abelian chief factor of K must involve a prime other than the characteristic, and for each such prime p the elementary abelian p-factors of G have rank at most $n! + n$, for example. Finally the number of non-abelian chief factors of K is bounded in terms of n only by Kargapolov's lemma again [8, 9.29] and consequently the intersection of the centralizers in K of the non-abelian chief factors of K is soluble, and its index in K is bounded by an integer t that depends upon n and $(H : S)$ only. We choose t monotonic in $(H:S)$.

The integer t above we carry through the rest of the proof.

Assume G is finitely generated. Let T be the unique maximal soluble normal subgroup of K [8, 3.8]. We prove that $(K : T) \leqslant t$. If not choose a set X of coset representatives of T in K with $|X| = t + 1$. If ϕ is a homorphism of G into a finite linear group of degree n then by the above case and the monotonic choice of t there exist $x, y \in X$ such that $x \neq y$ and yet $\langle x^{-1}y\rangle^K \phi$ is soluble. The x and y depend on ϕ, of course. But G is super-residually finite linear of degree n [8, 4.2], so there exists a set Φ of ϕ as above and a pair of elements $x \neq y$ of X such that $\bigcap_\Phi \ker \phi = \langle 1 \rangle$ and $\langle x^{-1}y\rangle^K \phi$ is soluble for every $\phi \in \Phi$. For if not for each pair $x \neq y$ of X there exists $k_{\{x,y\}} \in K \backslash \langle 1 \rangle$ such that whenever ϕ is as above with $\langle x^{-1}y\rangle^K \phi$ soluble we have $k_{\{x,y\}}\phi = 1$. But by [8, 4.2] and the finiteness of X there exists ϕ with $k_{\{x,y\}}\phi \neq 1$ for all $\{x, y\}$. This contradiction proves the existence of Φ. By [8, 3.7], for example $\langle x^{-1}y\rangle^K$ is soluble so $\langle x^{-1}y\rangle^K \subseteq T$. This contradiction of the choice of X completes this part of the proof.

The general case. Let X be any finitely generated subgroup of K. Then there exists a finitely generated subgroup G_1 of G such that if $H_1 = G_1 \cap H$ and $K_1 = H_1^{G_1}$, then $X \subseteq K_1$. By the previous case and the choice of t there is a soluble normal subgroup T_1 of K_1 with $(K_1 : T_1) \leqslant t$. Thus X contains a soluble normal subgroup $Y = X \cap T_1$ with $(X : Y) \leqslant t$. This is for any such X and therefore K contains a (locally) soluble normal subgroup T with $(K : T) \leqslant t$.

6. Finitely Generated Groups

6.1. *Let G be a group with a nilpotent normal subgroup N such that G/N is abelian by finite. Let H be a subgroup of G such that HN/N is finitely generated. If $G \mathbf{w} H$ then H* asc *G, while if $G \,|\mathbf{w}\, H$ then H* sn *G.*

Proof. By 3.2 we have $HN \operatorname{asc} G$ in the first case and $HN \operatorname{sn} G$ in the second, so we may assume that $G = HN$. We induct on the nilpotency class of N. Let Z be the centre of N. By induction $HZ \operatorname{asc} G$ (resp. $HZ \operatorname{sn} G$). We assume therefore that $G = HZ$, where $H/C_H(Z)$ has a finitely generated abelian normal subgroup $K/C_H(Z)$ of finite index. Factor out by $C_H(Z/(H \cap Z))$. Thus assume $C_H(Z) = \langle 1 \rangle$, so in particular $H \cap Z = \langle 1 \rangle$.
If $k \in K$ then $\langle k \rangle Z$ is normal in KZ. If $G \mathbf{w} H$ then $\langle k \rangle Z$ is a hypercentral group and if $G \,|\mathbf{w}\, H$ then $\langle k \rangle Z$ is nilpotent. Thus KZ is hypercentral in the first case [7, Vol. 1, p. 51] and nilpotent in the second [7, Vol. 1, p. 49]. If $G \mathbf{w} H$ then clearly each upper K-central factor of Z is hypercentral as H-module, while if $G \,|\mathbf{w}\, H$ each upper K-central factor of Z has finite H-central height by 2.1. The result follows.

6.2 Theorem. *Let F be a field, let G be a subgroup of $GL(n, F)$ and let H be a soluble-by-finite subgroup of G. Suppose there exists a finitely generated subring R of F with $H \subseteq GL(n, R)$.*

(a) *If $G \mathbf{w} H$ then H* asc *G.*

(b) *If $G \,|\mathbf{w}\, H$ then H* sn *G.*

(c) *If $G \mathbf{w}_m H$ then $H \lhd^d G$, where d is determined by m, n and R only.*

Theorem 1.3 follows immediately from 6.2.

Proof. By 5.1 the normal closure H^G is soluble by finite in all cases. Let U be the unipotent radical of H^G. Then H^G/U is abelian by finite and HU/U is finitely generated (cf. [8, 4.10]). Thus parts (a) and (b) follow from 6.1.
Now let $G \mathbf{w}_m H$. We may assume by 5.1 that G is soluble by finite such that, using Mal'cev's Theorem, G has a closed triangularizable normal subgroup T of finite index bounded in terms of n and $(H : S)$, only, where S is the maximal soluble normal subgroup of H. By Proposition 6 of [12] $(H:S)$ is bounded in terms of n and R only. By Wielandt's Theorem $HT \operatorname{sn} G$.
Let U be the unipotent radical of T, let Z be the centre of U and set $K = H \cap T$. We have seen that $(H : K) \leqslant (G : T)$ is bounded. Also by [12, Proposition 6] the minimal number f of generators of the abelian group $K/(K \cap U)$ is bounded in terms of n and R. By 2.3 we have

$$[Z, {}_{fm}K] \subseteq K \cap Z = H \cap Z.$$

Thus we deduce that the subnormal depth of H in HZ is bounded by applying 2.1 to the action of H/K on each of $[Z, {}_iK](H \cap Z)/[Z, {}_{i+1}K](H \cap Z)$. An elementary induction on the class of U shows the depth of H in HU is bounded and a further application of 2.1, to the action of H/K on $T/(T \cap HU)$, completes the proof.

7. Rational Groups

7.1. *Let F be a finite extension field of $\mathbb{Q}$. Let $X \subseteq Y \subseteq F$ be additive subgroups of F and let $a \neq 1$ be an element of F such that $Ya \subseteq Y$ and $Y(a-1)^m \subseteq X$ for some integer m. Then Y/X is finite.*

Proof. Since $(F : \mathbb{Q}) < \infty$ there exists an ireducible polynomial $f(T)$ over $\mathbb{Q}$ with $f(a) = 0$. We choose f with $f \in \mathbb{Z}[T]$ (but not necessarily monic). The polynomials $T-1$ and $f(T)$ are coprime since $a \neq 1$. Thus there exist h, $k \in \mathbb{Z}[T]$ and a positive integer d such that

$$d = (T-1)\, h(T) + f(T)\, k(T).$$

Then $dY(a-1)^i \subseteq Y(a-1)^{i+1}$ for each i and $d^m Y \subseteq X$. But Y has finite rank at most $(F : \mathbb{Q})$, so Y/X is finite of order dividing $d^{m(F:\mathbb{Q})}$.

7.2 *Proof of* 1.4. We may replace G by H^G, so by 5.1 we may assume that G is soluble by finite. Then G has a triangularizable normal subgroup T of finite index. Let U be the unipotent part of T and Z the centre of U. Now U can be unitriangularized over $\mathbb{Q}$ [8, 1.21], so U has finite rank at most $\frac{1}{2}n(n-1)$.

(a) By 3.5 we have $H \operatorname{asc} HZ$. Induction on the class of U yields. $H \operatorname{asc} HU$. By 3.2 we have $HU/U \operatorname{asc} G/U$ and so $H \operatorname{asc} G$.

(b) T is triangularizable over some finite extension field F of $\mathbb{Q}$. Then U contains a central series $\{Z_i\}$ of finite length of normal subgroups of T such that for each i the group Z_i/Z_{i-1} is isomorphic to an additive subgroup of F, the group $T/C_T(Z_i/Z_{i-1})$ is isomorphic to a multiplicative subgroup of F and the action of T on Z_i/Z_{i-1} is given by multiplication in F. With $K = H \cap T$ we have $K \cap Z_1$ normal in KZ_1 and for each $k \in K$ there exists m with $[Z_1, {}_mk] \subseteq K \cap Z_1$. If $[Z_1, K] \neq \langle 1 \rangle$ then by 7.1 the group $Z_1/(K \cap Z_1)$ is finite and 3.1 shows that $K \operatorname{sn} KZ_1$. Induction yields $K \operatorname{sn} KU$. Thus there exists an integer e with $[Z, {}_eK] \subseteq H \cap Z$; recall Z is the centre of U. Then 2.1 applied to the action of H/K on each of

$$[Z, {}_iK](H \cap Z)/[Z, {}_{i+1}K](H \cap Z)$$

shows that $[Z, {}_fH] \subseteq H \cap Z$ for some integer f and so $H \operatorname{sn} HZ$. The usual induction yields $H \operatorname{sn} HU$ and 3.2 applied to G/H completes the proof.

(c) By [13, Proposition 2, Corollary 1] we can choose T such that $(G:T)$ is bounded in terms of n only. Also $(H:K) \leqslant (G:T)$; $K = H \cap T$ as before. Set

$$l = (H:K)((H:K) - 1)(m - 1).$$

By 2.1 for each i we have

$$[Z, {}_iK, {}_lH] \subseteq [Z, {}_{i+1}K](H \cap Z).$$

Thus by 2.2 we have

$$[Z, {}_{2^{m-1}l}H]^{m^*} \subseteq H \cap Z.$$

We have seen that U has bounded rank, so

$$X = [Z, {}_{2^{m-1}l}H](H \cap Z)/(H \cap Z).$$

has finite order x say, bounded in terms of m and n. Now $H \operatorname{sn} G$ by Part (b), so X is H-hypercentral and $[X, {}_xH] = \langle 1 \rangle$. Consequently

$$[Z, {}_{2^{m-1}l+x}H] \subseteq H \cap Z.$$

Thus $H \lhd^{2^{m-1}l+x} HZ$ and the usual induction bounds the subnormal depth of H in HU. By 2.1 again $HU \lhd^l HT$ and $HT \lhd^{(G:T)} G$ trivially. The proof is complete.

7.3 *Proof of* 1.5 *Parts* (b) *and* (c). Here $G \in \mathfrak{S}_1\mathfrak{F}$ and at least $G \,|\, \mathbf{w}\, H$. The group G contains a normal subgroup A such that A is a direct product of a finite number, say s, of Prüfer groups and such that G/A is isomorphic for some n to a subgroup of $GL(n, \mathbb{Q})$ [7, Vol. 2, pp. 131, 137; 10, 1.1]. Repeated use of 3.4 yields $H \lhd^s HA$ and the result follows from 1.4.

8. Examples

8.1. As Peng [5] points out, if W is the wreath product of a cyclic group of prime order p by a countable infinite, elementary abelian p-group K then $W \mathbf{w}_p K$ and yet $K = N_W(K)$. If G is a free group of countable rank map G homomorphically onto W and let H be the inverse image of K in G. Then $G \mathbf{w}_p H$ and $H = N_G(H)$. But G is isomorphic to a linear group of degree 2 over quite small rings, for example $\mathbb{Z}$; cf. [8, 2.8, 2.9 and Exercise 2.3].

8.2. Repeat the construction of 8.1 using for G a free metabelian group of countable rank. This G is also isomorphic to a linear group of degree 2 [8, 2.11] but over a much larger ring, e.g., $\mathbb{R}$. To obtain a soluble example in characteristic $p > 0$, replace G by a free group of countable rank in the variety generated by the two words

$$[x, y]^p \quad \text{and} \quad [[w, x], [y, z]].$$

8.3. There exists a finitely generated Engel group E that is not nilpotent (Golod [1]). By [7, 6.11] there is a cyclic non-ascendent subgroup L of E. Let G be a free group of *finite* rank the minimal number of generators of E and let H be the inverse image of L in G under some homomorphism of G onto E. Then $G \mathbf{w} H$ but H is not ascendant in G.

8.4. Let

$$G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : \alpha \text{ a 2th power root of 1 in } \mathbb{C} \right\rangle$$

and

$$H = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

G is periodic, being an infinite locally dihedral group. Here $G \mathbf{w} H$ and $H \operatorname{asc} G$ but H is not subnormal in G. The group $\mathbb{C}_{p^\infty} \wr \mathbb{C}_p$ affords a similar example for any prime p; see [5, p. 230].

8.5. Let

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq GL(2, \mathbb{Q})$$

and

$$H = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Then G is metabelian, $H \operatorname{asc} G$ but H is not subnormal in G. For if

$$A = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

and $B = H \cdot A^G$ then $A \lhd B \lhd G$ and B/A is an infinite locally dihedral group. Thus 8.5 follows from 8.4.

8.6. If p_i is the ith prime let A_i be cyclic of order p_i^i and let B_i be the Sylow p_i-subgroup of $\operatorname{Aut} A_i$. If G_i is the split extension of A_i by B_i then B_i is subnormal in G_i of depth i exactly. Now let G be the direct product of all the G_i and set $H = \langle B_i : i = 1, 2, \ldots \rangle$. Then G is a metabelian group of finite rank and H is a subgroup of G with $G \mid \mathbf{w}\, H$ and $H \operatorname{asc} G$ such that H is not subnormal in G.

8.7. There exists a countable metabelian linear group of degree 2 with a non-subnormal subgroup H such that $G\, \mathbf{w}_2\, H$ and $H \operatorname{asc} G$. For let $W_i = C_i \wr K_i$, where C_i has order 2 and K_i is elementary abelian of order 2^i. Let W be the direct product of the W_i over $i = 1, 2, \ldots$. Choose a free metabelian group G mapping homomorphically onto W and let H be the inverse image of $\langle K_i : i = 1, 2, \ldots \rangle$.

8.8. Let

$$a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where x is *not* a root of unity. Set $G = \langle a, b \rangle$, G is an infinite dihedral group. For all $i \geqslant 1$, if $H_i = \langle a^{2^i}, b \rangle$, then $G \mid \mathbf{w}\, H_i$ and $H_i \ntriangleleft^{i-1} G$. To construct an example with fixed H note that all the H_i above are isomorphic. Thus let $G_i = \langle G, \operatorname{diag}(x_i, x_i^{-1}) \rangle$, where $x_i^{2^i} = x$ and set $H = G$. Then $G_i \mid \mathbf{w}\, H$ and $H \ntriangleleft^{i-1} G_i$ for all $i \geqslant 1$.

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